Confidence-based Contractor, Propagation and Potential Cloud for Differential Equations

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Introduction

An interval aims to bound all the values of an uncertain variable, for example provided by a measurement device [5]. This approach is highly effective for every safety, verification or validation procedures because the intervals are conservative. The major inconvenience is that intervals are sometimes too pessimistic. Otherwise, it is conceivable that a measure can be associated to guaranteed bounds, an interval, and also a confidence level coming from past observations.

A novel contractor is proposed to filter an interval following a confidence level given on the associated quantity. This confidence level is an input of the contractor, the "new" information, while the probability distribution of the considered variable is a characteristic of the associated random variable.

Combining intervals and a probability has been already proposed in numerous papers using techniques such as p-boxes [8], fuzzy sets [4], box-particles [1] and potential cloud [7].

We are particularly interested in Ordinary Differential Equations (ODEs) and validated methods to compute their reachable sets via validated simulation [6, 3]. In the case of Initial Value Problems (IVPs) with ODEs, the initial state is primordial. An uncertain initial state is generally bounded in a box. As experimentation, we propose to consider in addition to this initial box some confidence levels, and we apply the presented approach. It allows us to describe the reachable set by a cloud. This richer result can then be used in different control problems, parameter synthesis, verification, etc.

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Preliminaries

When focusing on symmetric distributions such as the normal distribution, one can define:

Confidence interval is a set $\mathcal{S}$ for which the probability of the given random variable to be in this set is equal to the given probability P.

Probability density function is most commonly associated with absolutely continuous univariate distributions. A random variable X has density f_X , where f_X is a non-negative Lebesgue-integrable function, if:

$$P = \Pr[a \leq X \leq b] = \int_a^b f_X(x)\, dx.$$

Confidence level (e.g. $CL = 95\%$) allows to define the corresponding confidence interval (e.g. $C_{95\%}$). This interval can be obtained by observation (statistical approach) or with the help of a known distribution (probability approach). A new measure $\hat{x}$ coming from the (same) experiment will be in the associated confidence interval such that:

$$\hat{x} \in C_{95\%} \quad 95\% \text{ of the time.}$$

Confidence-based Contractor

We propose the following confidence-based contractor:

$$\begin{aligned} Cbc([x]|f_X, cc) : \quad & \mathbb{IR} && \mapsto \mathbb{IR} \\ & [x] && \rightarrow [x] \cap [y] \end{aligned}$$

with $[y]$ the confidence interval defined such that

$$\Pr[x \in [y]] = \int_{[y]} f_X(x)\, dx = cc,$$

cc being the confidence coefficient ($0 \leq cc \leq 1$). For example, one can use the parameter assignment $cc = 0.68$ for a confidence level of 68%.

Figure 1 illustrates the effect of the confidence-based contractor applied to the following example. Let X be a random variable with a normal distribution, such that

$$f_X(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

with $\mu = 1.0$ and $\sigma = 1.0$. The quantity X is observed and one measure is obtained: $[x] = [0.7, 2.1]$. A confidence level of 68.27% is given on X, that is to say that we are confident on the accuracy of the observations, so X stays close to its mean. Our method computes the contraction such that:

$$\begin{aligned} Cbc([0.7, 2.1] \mid f_X, 0.6827) &= [0.7, 2.1] \cap [0, 2] \\ &= [0.7, 2.0] \end{aligned}$$

So upper bound is reduced with respect to the confidence level. The pessimism induced by interval approach is then limited.
Two special cases can be described:

- $\forall [x], Cbc([x]|f_X, 0) = \emptyset$ (annihilating element)

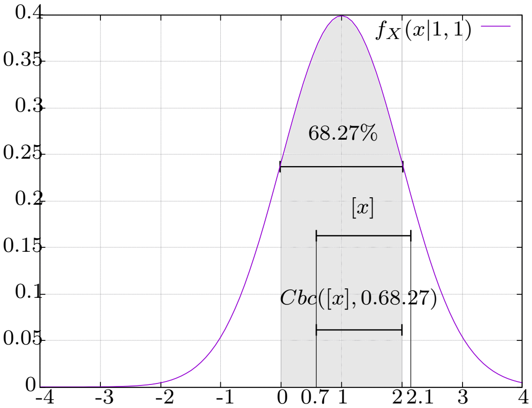

Figure 1: Illustration of confidence-based contraction.

- $\forall [x], Cbc([x]|f_X, 1) = [x]$ (identity element)

For two different confidence coefficients cc_1 and cc_2 such that $cc_1 < cc_2$, the following order holds:

$$\forall [x], Cbc([x]|f_X, cc_1) \subset Cbc([x]|f_X, cc_2)$$

The contractor Cbc can be composed with other contractors or with itself.

The confidence-based contractor presented in this paper needs the computation of the confidence interval associated to a given confidence level. Three cases can be detailed:

Case 1: a well known probability distribution and a particular confidence level with known confidence interval. For example, a normal distribution with a 95% confidence level gives a confidence interval $[\mu - 2\sigma, \mu + 2\sigma]$.

Case 2: a probability distribution with a known inverse function, such as the inverse of error function for Gaussian density function (*i.e.* erf^{-1}).

Case 3: the general case without any particular value. For this case, a predictor-corrector algorithm exploiting the symmetry of the distribution is designed.

Reachability and Potential Clouds

Computing the reachable set of an initial value problem defined as follows:

$$\begin{cases} \dot{\mathbf{y}}(t) = g(t, \mathbf{y}(t)) \\ \mathbf{y}(0) \in [\mathbf{y}_0] \subseteq \mathbb{R}^n. \end{cases} \quad (1)$$

can be performed with validated simulation tools. It provides a tube enclosing $\mathbf{y}(t; [\mathbf{y}_0])$ for $t \in [0, T]$. The confidence contractor presented here can be used to filter the initial states w.r.t. different confidence levels. The contraction can be propagated along the tube[2] to build a potential cloud on the final state. This approach is useful in the case of differential constraint while a constraint can be verified for some confidence levels but not for the global intial set ($\mathbf{y}(0) \in [0, 2]$). An example is given in Fig. 2. With a confidence level of 50% on initial states, the constraint $[y(5; [y_0])] \subset [1.5, 1.6]$ is verified.

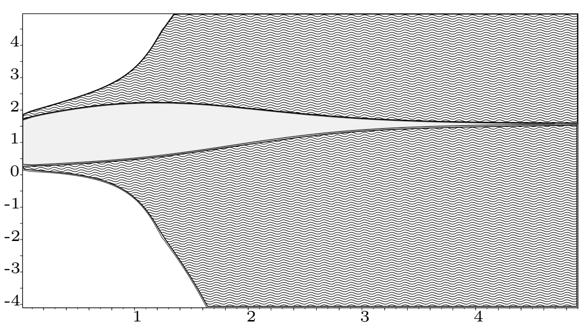

Figure 2: Validated trajectories for 60% (in grey waves) and for 50% (in light grey) confidence contraction.

Conclusion

A novel interval contractor based on the confidence assigned to a random variable was proposed. It makes possible to consider at the same time an interval in which the quantity is guaranteed to be, and a confidence level to reduce the pessimism induced by interval approach. As application, we proposed to compute the reachable set of an ordinary differential equation under the form of a potential cloud, with respect to confidence levels on initial value.

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