Analyzing Uncertain Dynamical Systems After State-Space Transformations Into Cooperative Forms

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Introduction

There are different reasons for the occurrence of uncertainty. It can appear due to model simplifications, approximation of nonlinearities, imprecise parameter knowledge and/or order reduction as well as physical and numerical restrictions of the system itself. Uncertainty caused by measurement noise and sensor inaccuracies are further examples. In any case, uncertainties can be treated either stochastically or as bounded quantities in terms of worst case scenarios, where the lower and upper bounds are summarized in an interval. Hence, interval arithmetic is a common tool, see [3]. Unfortunately, its use tends to lead to overestimation due to the so-called wrapping effect. To avoid this, cooperativity has already been investigated in several papers, [2, 4, 5]. A system is cooperative, if for an autonomous dynamic system

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t)\right) \ , \quad \mathbf{x} \in \mathbb{R}^n \ , \tag{1}$$

all off-diagonal elements $J_{i,j}$, $i, j \in \{1, \ldots, n\}$, $i \neq j$, of the corresponding Jacobian

$$\mathbf{J} = \frac{\partial \mathbf{f}\left(\mathbf{x}\right)}{\partial \mathbf{x}} \tag{2}$$

are strictly non-negative according to

$$J_{i,j} \geq 0 \ , \quad i, j \in \{1, \ldots, n\} \ , \quad i \neq j \ . \tag{3}$$

This means, that state trajectories $\mathbf{x}(t)$ starting in the positive orthant $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n \mid x_i \geq 0, \forall i \in \{1, \ldots, n\}\}$ are guaranteed to stay in this positive orthant for all $t \geq 0$ because $\dot{x}_i(t) = f_i\left(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots x_n\right) \geq 0$ holds for all components $i \in \{1, \ldots, n\}$ of the state vector as soon as the state variable x_i reaches the value $x_i = 0$. The advantage of cooperativity is the simplification of several tasks such as the computation of guaranteed state enclosures, the design of interval observers, forecasting worst-case bounds for selected system outputs in predictive control and the identification of unknown parameters.

Main Idea

Many system models in biological, chemical, and medical applications are naturally cooperative. However, there is also a great number of systems (typically from the fields of electric, magnetic, and mechanical applications) which do not show this property if the state equations are derived using first-principle techniques. Hence, it is often desired to transform such system models into an equivalent cooperative form. If a system

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t)\right) \tag{4}$$

is linear, it can be given in the state-space representation

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{p}) \cdot \mathbf{x} + \mathbf{B}(\mathbf{p}) \cdot \mathbf{u} \tag{5}$$

with the state vector $\mathbf{x}$ and the input $\mathbf{u}$ considering parameter uncertainty in the elements of the system matrix $\mathbf{A}(\mathbf{p})$ as well as the input

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matrix $\mathbf{B}(\mathbf{p})$. Moreover, most nonlinear systems can be reformulated into a quasi-linear state-space representation

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x}) \cdot \mathbf{x} + \mathbf{B}(\mathbf{x}) \cdot \mathbf{u} \ , \tag{6}$$

where the uncertainty lies in the state dependencies due to nonlinear expressions in the right-hand sides of (4). Both representations (5) and (6) describe uncertain systems, which can be transformed into cooperative forms by means of Eqs. (1)–(3). If the system model is controllable (or at least stabilizable) and the desired operating state is set to $\mathbf{x} = \mathbf{x}_\mathrm{s} = \mathbf{0}$ without loss of generality for the steady-state input $\mathbf{u} = \mathbf{u}_\mathrm{s} = \mathbf{0}$, a feedback controller is introduced in Eqs. (5) and (6) according to $\mathbf{u} = -\mathbf{K}(\mathbf{p}) \cdot \mathbf{x}$ or $\mathbf{u} = -\mathbf{K}(\mathbf{x}) \cdot \mathbf{x}$, respectively, leading to the following state-space representations:

$$\dot{\mathbf{x}} = (\mathbf{A}(\mathbf{p}) - \mathbf{B}(\mathbf{p})\mathbf{K}(\mathbf{p}))\cdot\mathbf{x} = \mathbf{A}_\mathrm{C}(\mathbf{p}) \cdot \mathbf{x} \tag{7}$$

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For the transformation into an equivalent cooperative form, we make use of a method developed in [2] for linear systems with crisp parameterization. This approach has been extended to uncertain systems in [5] and generalized in [4] to cover real-life applications in an efficient manner. It was shown that one needs to distinguish between systems with purely real and conjugate complex eigenvalues. For the presented paper, we will concentrate on the former. It was assumed that the uncertain system matrix can be expressed by the element-wise defined inequality

$$\mathbf{Z}_\mathrm{a} - \boldsymbol{\Delta} \leq \mathbf{Z} := \mathbf{A}_\mathrm{C} \leq \mathbf{Z}_\mathrm{a} + \boldsymbol{\Delta} \ , \tag{9}$$

where $\boldsymbol{\Delta}$ consists of the (symmetric) worst-case bounds of all entries in $[\mathbf{A}_\mathrm{C}]$. Note, the midpoint matrix $\mathbf{Z}_\mathrm{a} = \mathbf{Z}_\mathrm{a}^T$ in Eq. (9) is assumed to be symmetric in what follows. A Metzler matrix $\mathbf{R} = \mu\mathbf{E}_n - \boldsymbol{\Gamma}$ is searched for, which has the same eigenvalues as $\mathbf{Z}_\mathrm{a}$, with a constant $\mu \in \mathbb{R}$ and a diagonal matrix $\boldsymbol{\Gamma} \in \mathbb{R}^{n\times n}$; $\mathbf{E}_n \in \mathbb{R}^{n\times n}$ is a matrix with all elements equal to 1 and $\boldsymbol{\Gamma} = \rho\mathbf{I}_n$ with $\rho > \mu$ and the identity matrix $\mathbf{I}$ of order n. If $\mathrm{eig}(\mathbf{R}) = \mathrm{eig}(\mathbf{Z}_\mathrm{a})$, according to [2], there exists an orthogonal matrix $\mathbf{S} \in \mathbb{R}^{n\times n}$ such that $\mathbf{S}^T\mathbf{Z}\mathbf{S}$, respectively, is Metzler provided that $\mu > n||\boldsymbol{\Delta}||_\mathrm{max}$, where $||\boldsymbol{\Delta}||_\mathrm{max}$ denotes the maximum absolute value of $\boldsymbol{\Delta}$. However, in several practical cases finding the transformation matrix $\mathbf{S}$ is not trivial. Thus, this approach was converted into a computationally feasible optimization problem formulated with linear matrix inequality (LMI) constraints [1]. This is done with the main goal of a generalization to cover both possible uncertainties of Eqs. (5) and (6). Both types of system models with time- and state-dependent parameter uncertainties are investigated for real-life electric RLC-circuits.

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