Efficient computation of the set of stabilizing controllers for an LTI System using intervals

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Introduction

Guaranteed characterisation of the set of controllers stabilizing a system is a major problem in control theory. Interval analysis gives a tool to solve this problem as described in [2]. However it is computationally expensive for high order systems (>7) with a lot of controller gains (>5) or parametric uncertainties. This work deals with author attempts on alternative approaches to improve this computation efficiency. Section 2 recalls stability criteria for systems and how they are used by an interval analysis algorithm in robust control. Section 3 gives some alternative implementations of stability criteria. Section 4 suggests a different algorithm.

Stabilizing Controllers Set Computation

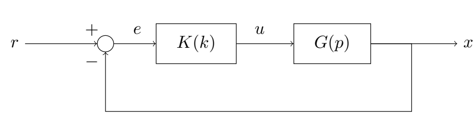

Figure 1: Closed-Loop System F

Let $G(p)$ and $K(k)$ be Linear Time Invariant Systems (LTI). G is called the regulated systems, p are the uncertain parameters of G. K is called the controller and k are the controller gains. $G(p)$ and $K(k)$ are linked in a closed-loop system $F(p,k)$ as shown in Fig. 1. $F(p,k)$ is also an LTI system. The problem stated in this work is to find the set $\mathcal{K}_{stable}$ stabilizing F for all values of p inside a given set $\mathcal{P}$.

Given a state-space representation of F, $(A_F(p,k), B_F, C_F, D_F)$, An internal stability criterion for F is given by the Routh-Hurwitz criterion [2] : Given $P(p,k,s) = \det(sI - A_F(p,k))$ the characteristic polynomial of A_F and $a_i(p,k) \in \mathcal{R}^n$ its coefficients and H the Hurwitz Matrix given by:

$$H = \begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}$$

F is internally stable iff all the minors of H are strictly negative.

As those minors have an analytic expression, this criterion is available for a set computation via interval analysis. [2] translates the stabilizing set finding problem as a constraint satisfaction problem (CSP) and provides an algorithm to solve it by operating dichotomies on an initial box of values of k. However the complexity of this algorithm is exponential with the dimension of the interval box k and the evaluation pessimism increases dramatically with the order n of the system F for a naive implementation of the Routh-Hurwitz stability criterion.

Alternative Stability Criteria

Following the previous statement, several solutions are explored to control the computational complexity.

The first improvement is to use the Lienard-Chipart criterion [4] which is a direct derivative from the Routh-Hurwitz. However it is more efficient as it tests only half the minors of the Hurwitz matrix, giving the opportunity to not compute the minor with the highest degree, which suffer the most from evaluation pessimism. For the same computational complexity, it is possible to deal with systems with one more order.

The second tested solution is to improve the Directed Acyclic Graph (DAG) of the stability criterion expression for a more precise interval evaluation. For that, the operator $HurwStab([a_i])$ is created at a low implementation level for interval computation. This results in a significant reduction in evaluation pessimism.

The last one is to test an alternative criterion based on the Argument Principle [5] formula: given a complex function f and a complex positively oriented contour $\oint_C$ where f never equals zero,

$$\oint_C \frac{f'(z)}{f(z)}\, dz = 2\pi i(Z - Q)$$

where Z and Q are respectively the number of zeros and poles of f. The idea is to replace f by the characteristic polynomial (which does not have poles) to test if it has roots on the right half plane which cause instability. It is possible with a clever contour like on Fig. 2, with a maximum radius fixed with Gershgorin circles. It provides an alternative criterion as long as the algorithm can compute integrals with interval analysis.

Despite the addition of complexity and pessimism caused by integral computation, this alternative criterion seems interesting for some problems with high order systems.

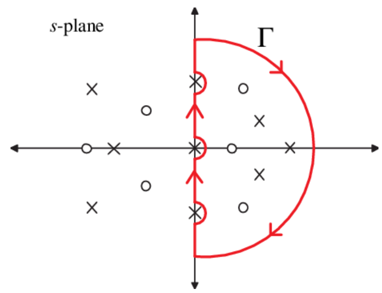

Figure 2: A Nyquist Contour Γ

Alternative Set Computation Algorithm

Eventually, an alternative algorithm is suggested to compute the stabilizing set. It uses Kharitonov theorem [3]. Kharitonov states that, for a characteristic polynomial with interval coefficients, it is sufficient to test only four polynomial edges to prove complete stability of all polynomials in the set.

As far as the author knows, there is no method to prove complete instability for such interval polynomials. However, Dabbene [1] gives a fast randomized algorithm to find a stable punctual polynomial inside interval polynomials. A failure of the Dabbene algorithm suggests a complete instability of the set. For a polynomial order < 14, about 1000 iterations seem to provide a reliable result.

Here, Kharitonov and Dabbene are seen as operators taking interval polynomial coefficients and returning a Boolean (Kharitonov is true if the set is stable, Dabbene is true if it found a stable point in the set). Based on definitions given in previous sections, the author defines two operators:

- The coefficient operator :

$$\begin{gathered}(p,k) \rightarrow a_i(p,k)\\ p \in \mathcal{P}, k \in \mathcal{K}, a_i \in \mathcal{R}^n\end{gathered} \tag{1}$$

- The Dabbene-Kharitonov (DK)-operator :

$$[a_i] \rightarrow \begin{cases} true & if \quad Kharitonov([a_i]) \\ false & if \quad \neg Dabbene([a_i]) \\ unknown & if \quad Dabbene([a_i]) \end{cases} \tag{2}$$

Using those operators, the alternative Set Computation algorithm steps are as follows:

1. With a given set of values for (p,k) the

coefficient operator provides a set of polynomial coefficients $[a_i]$.

2. The DK-operator is used by a paver to provide the set of stable polynomials given by their coefficients.

3. Based on the result of 2., a Set Inversion Algorithm returns the stabilizing controller gains set $\mathcal{K}_{stable}$.

This algorithm is not guaranteed as the Dabbene test is not. However still it seems relevant because a failure of the Dabbene test is unlikely to occur as it is explained in the statistical analysis provided in [1]. This algorithm could be efficient insofar as the KD-operator does not introduce evaluation pessimism. It is not the case for the coefficient operator but its expression is assumed to be simple regarding Routh-Hurwitz criterion expression.

A discussion on the complete implementation of this algorithm will conclude the work.

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