Computation of integrals with interval endpoints

Olivier Mullier*[1] and Julien Alexandre dit Sandretto[1]

[1]ENSTA ParisTech, 828 Boulevard des maréchaux, 91120 Palaiseau
[1]{mullier, alexandre}@ensta.fr

Keywords: Integral; Set-membership Computation; Interval Methods.

Introduction

Numerical integration is one of the fundamental tool of scientific computation. Providing a reliable result to such problem is important for validated simulation [1] or for global optimization with a continuous objective function [3]. An important work on inclusion methods for integral equations can be found in [2]. In our presentation, we propose an efficient guaranteed method for the computation of the integral of a nonlinear continuous function f between two interval endpoints $[x_1]$ and $[x_2]$, we call interval integrals:

Definition 1 (Interval integral)**.** Let $f : \mathbb{R} \to \mathbb{R}$, a continuous function and $[x_1]$, $[x_2] \in \mathbb{IR}$ two intervals. The interval integral of f with $[x_1]$ and $[x_2]$ as endpoints is denoted $\int_{[x_1]}^{[x_2]} f(x)dx$ and corresponds to the set

$$\int_{[x_1]}^{[x_2]} f(x)dx = \left\{ \int_{x_1}^{x_2} f(x)dx \;\middle|\; \begin{array}{l} x_1 \in [x_1] \\ x_2 \in [x_2] \end{array} \right\}. \quad (1)$$

This set considers all the integrals with the endpoints taken in the intervals $[x_1]$ and $[x_2]$. Three cases can occur whether the interval endpoints $[x_1]$ and $[x_2]$ are disjoint, intersect or one is included in the other.

The endpoints are disjoint As introduced in [2], an interval integral as defined in Definition 1 where the endpoints are disjoint can be decomposed as follows

$$\begin{aligned} \int_{[x_1]}^{[x_2]} f(x)dx &= \int_{[x_1]}^{\overline{x_1}} f(x)dx \\ &+ \int_{\overline{x_1}}^{\underline{x_2}} f(x)dx \\ &+ \int_{\underline{x_2}}^{[x_2]} f(x)dx. \end{aligned} \quad (2)$$

The endpoints intersect The interval integral in Eq. (1) can be subdivised with

$$\int_{[x_1]}^{[x_2]} f(x)dx = \int_{[\underline{x_1},\underline{x_2}]}^{[x_2]} f(x)dx$$
$$\bigcup \int_{[\underline{x_2},\overline{x_1}]}^{[\underline{x_2},\overline{x_1}]} f(x)dx \bigcup \int_{[\underline{x_2},\overline{x_1}]}^{[\overline{x_1},\overline{x_2}]} f(x)dx. \quad (3)$$

The first and the last interval integrals in the right member of Eq. (3) are of the same type as the one where endpoints are disjoint except that the integral can be equal to 0 when taking both the same endpoints.

One endpoint is included in the other When $[x_1] \subseteq [x_2]$, we have $\underline{x_2} \leqslant \underline{x_1} \leqslant \overline{x_1} \leqslant \overline{x_2}$ and the same decomposition as in Eq. (3) is possible:

$$\int_{[x_1]}^{[x_2]} f(x)dx = \int_{[x_1]}^{[\underline{x_2},\underline{x_1}]} f(x)dx,$$
$$\bigcup \int_{[x_1]}^{[x_1]} f(x)dx, \bigcup \int_{[x_1]}^{[\overline{x_1},\overline{x_2}]} f(x)dx \quad (4)$$

so we go back to the already treated kind of interval integral that occurred in the previous cases.

We see that in all cases, only three interval integrals occur:

$$\int_{[x]}^{\overline{x}} f(x)dx; \int_{\underline{x}}^{[x]} f(x)dx; \int_{[x]}^{[x]} f(x)dx. \quad (5)$$

*Corresponding author.

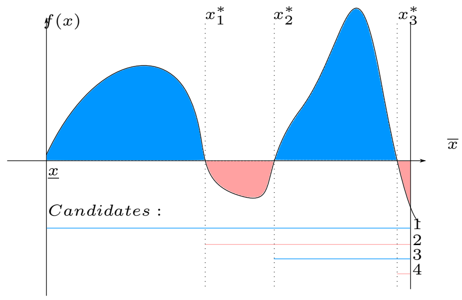

Figure 1: Example of computation of $\int_{[x]}^{\overline{x}} f(x)dx$ for $\mathcal{X}^* = \{x_1^*, x_2^*, x_3^*\}$ (blue: maximum; red: minimum).

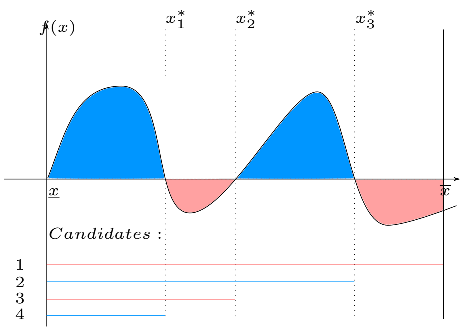

Figure 2: Example of computation of $\int_{\underline{x}}^{[x]} f(x)dx$ for $\mathcal{X}_2^* = \{x_1^*, x_2^*, x_3^*\}$.

Producing the minimum and the maximum of these interval integrals requires the parts where sub-integrals are positive and parts where they are negative. The change between positiveness and negativeness of the integral occurs at x such that $f(x) = 0$. Computing the minimum and maximum then requires to produce the set $\mathcal{X}^* = \{x \in [x] : f(x) = 0\}$. The minimum and the maximum candidates for all the interval integrals in Eq (5) can be defined using $\mathcal{X}^*$. When the arity of $\mathcal{X}^*$ is finite, the set of candidate to consider is then finite as well. Figure 1 provides an illustration of the candidates for $\int_{[x]}^{\overline{x}} f(x)dx$. In this case, we only have to consider 4 integral candidates to be the minimum and the maximum. In Figure 2, an illustration of $\int_{\underline{x}}^{[x]} f(x)dx$ is illustrated. The method for the computation of an interval integral consists in the computation of the set $\mathcal{X}^*$ and to find all the candidates to be the minimum and the maximum of the set described in Eq. (1). The method then provides the interval outer approximation of this set and also the endpoints at play.

Example We consider the computation of the interval integral $\int_0^{[0,1]} \frac{dx}{1+x^2}$. The result is:

$$\left[\int_0^0 \frac{dx}{1+x^2}, \int_0^1 \frac{dx}{1+x^2}\right] \subset [0, 0.78543] \quad (6)$$

The implementation of the computation of any interval integral is linear on the arity of $\mathcal{X}^*$ for $\int_{\underline{x}}^{[x]} f(x)dx$ and $\int_{[x]}^{\overline{x}} f(x)dx$. For $\int_{[x]}^{[x]} f(x)dx$, we need to consider the backward integrals as well since the first endpoint can be greater than the last one. The proposed algorithm is factorial on the arity of $\mathcal{X}^*$. This will be detailed in the presentation.

Acknowledgment

This research was supported of the "Chair Complex Systems Engineering - Ecole Polytechnique, THALES, DGA, FX, DASSAULT AVIATION, DCNS Research, ENSTA ParisTech, Télécom ParisTech, Fondation ParisTech and FDO ENSTA". This research is also partially funded by DGA MRIS.

References

[1] J. Alexandre dit Sandretto and A. Chapoutot. Validated explicit and implicit runge-kutta methods. *Reliable Computing electronic edition*, 22, 2016.

[2] G. F. Corliss. Computing narrow inclusions for definite integrals. *Computerarithmetic*, pages 150–179, 1987.

[3] E. Hansen. Global optimization using interval analysis—the multi-dimensional case. *Numerische Mathematik*, 34(3):247–270, 1980.