Toward the Development of Iteration Procedures for the Interval-Based Simulation of Fractional-Order Systems

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Introduction

In recent years, numerous interval-based simulation techniques have been developed which allow for a verified computation of outer interval enclosures for the sets of reachable states of dynamic systems represented by finite-dimensional sets of ordinary differential equations (ODEs). Here, especially the evaluation of IVPs is of interest, when both the systems' initial conditions and parameters can only be defined by finitely large domains, often represented by interval boxes. Suitable simulation techniques make use of series expansions of the solutions of IVPs with respect to time and (possibly) the uncertain initial conditions as well as of verified Runge-Kutta techniques. Solution sets are then typically represented by means of multi-dimensional intervals, zonotopes, ellipsoids, or Taylor models, cf. [5].

In most of these approaches, variants of the Picard iteration [1] are involved, which either determine the sets of possible solutions or at least worst-case outer enclosures with which time discretization errors are quantified. An example for a solution routine based entirely on this iteration is the exponential enclosure technique published in [9] and the references therein. It is applicable to systems with non-oscillatory and oscillatory behavior if the solution of the IVP of interest shows an asymptotically stable behavior. For non-oscillatory dynamics, the solution is determined by a real-valued iteration, while complex-valued interval analysis [7] is employed when eigenvalues with non-zero imaginary parts arise after a linearization of the state equations.

Although such enclosure techniques are well studied for IVPs of integer-order ODEs, the analysis of fractional-order differential equations (FDEs) has not yet received the same amount of interest. FDEs can be used efficiently in many engineering applications if the frequency response of a dynamic system is not characterized by variations of the amplitudes that consist of multiples of the slope $\pm 20\,\mathrm{dB}$ per frequency decade. The same holds for changes of the phase responses which do not coincide with integer multiples of $\pm\frac{\pi}{2}$, cf. [6,8]. In such cases, FDEs (for real-life applications often of the so-called Caputo type) can be used to significantly enhance modeling accuracy in comparison with integer-order ODEs. First extensions of the Picard iteration for determining interval enclosures to IVPs for FDEs were published in [4]. Practical applications where FDEs have significant advantages over classical ODEs can be found exemplarily in the field of modeling and state estimation for battery systems [10].

In the following, a brief summary of an extension of the exponential enclosure technique for FDEs published in [9] is given.

Interval Methods for FDEs

Consider the commensurate-order FDE system of Caputo type [6, 8]

$$\mathbf{x}^{(\nu)}(t) = \mathbf{A}\left(\mathbf{x}(t)\right) \cdot \mathbf{x}(t)\ , \quad \mathbf{x}(t) \in \mathbb{R}^n\ , \quad (1)$$

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with $0 < \nu < 1$, where initial conditions for $\mathbf{x}(t)$ at $t = 0$ are defined in analogy to classical IVPs for integer-order ODEs. Then, for the iteration step κ, (complex-valued) parameters $\lambda_i^{\langle\kappa\rangle}$ with $[\mathbf{\Lambda}]^{\langle\kappa\rangle} := \operatorname{diag}\left\{[\lambda_i]^{\langle\kappa\rangle}\right\}$, $i \in \{1, \ldots, n\}$, are determined in the ansatz

$$\mathbf{x}(t) \in [\mathbf{x}_e]^{\langle\kappa\rangle}(t) := \mathbf{E}_{\nu,1}\left([\mathbf{\Lambda}]^{\langle\kappa\rangle} \cdot t^{\nu}\right) \cdot [\mathbf{x}_e](0), \quad (2)$$

where $\mathbf{E}_{\nu,1}\left([\mathbf{\Lambda}]^{\langle\kappa\rangle} \cdot t^{\nu}\right)$ is a diagonal matrix with the element-wise evaluation of the Mittag-Leffler function [3] $E_{\alpha,\beta}(\zeta) = \sum\limits_{i=0}^{\infty} \frac{\zeta^i}{\Gamma(\alpha i+\beta)}$. According to [9], the parameters of the enclosures (2) are computed by

$$\begin{aligned} [\lambda_i]^{\langle\kappa+1\rangle} := {} & a_{ii}\left([\mathbf{x}_e]^{\langle\kappa\rangle}([t])\right) \\ & + \sum_{\substack{j=1 \\ i \neq j}}^{n} \left\{ a_{ij}\left([\mathbf{x}_e]^{\langle\kappa\rangle}([t])\right) \right. \\ & \left. \cdot \frac{E_{\nu,1}\left([\lambda_j]^{\langle\kappa\rangle} \cdot [t]^{\nu}\right)}{E_{\nu,1}\left([\lambda_i]^{\langle\kappa\rangle} \cdot [t]^{\nu}\right)} \cdot \frac{[x_{e,j}](0)}{[x_{e,i}](0)} \right\} . \end{aligned} \quad (3)$$

In this contribution, simulation results for verified IVP solutions for FDEs are presented in comparison with corresponding analytic solutions given, for example, in [2].

Finally, current research directions will be pointed out together with possible solution approaches for yet open problems resulting from the fact that FDE models represent memory effects over infinitely long time horizons.

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